Optimal Designs, Supplementary Difference Sets and Multipliers

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Abstract

We investigate multipliers of $2-\{v;\ q^2,\ q^2;\ \lambda\}$ supplementary differences sets where cyclotomy has been used to construct D-optimal designs.

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1 Introduction

Let $S_1,\ S_2,\ \cdots,\ S_u$ be subsets of V, a finite abelian group of order v written in additive notation containing $k_1,\ k_2,\ \cdots,\ k_u$ elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i. If T contains each non-zero element of V a fixed number of times λ say, then $S_1,\ S_2,\ \cdots,\ S_u$ will be called $u-\{v;\ k_1,\ k_2,\cdots,\ k_u;\ \lambda\}$ *supplementary difference sets*. Throughout this paper as in [7] where they were first defined and [8], [9] and [10] this will be abbreviated as sds.

The parameters of $u-\{v;\ k_1,\ k_2,\cdots,\ k_u;\ \lambda\}$ supplementary difference sets satisfy

$$\lambda(v-1)=\sum_{i=1}^{u}k_i(k_i-1). \tag{1}$$

If $k_1=k_2=\cdots=k_u$ we will write $u-\{v;\ k;\ \lambda\}$ to denote the u supplementary difference sets and (1) becomes

$$\lambda(v-1)=uk(k-1). \tag{2}$$

We shall be concerned with collections defined on a fixed group V of order v, in which repeated elements are counted multiply, rather than as with sets. If T_1 and T_2 are two collections then T_1 and T_2 will denote the result of adjoining the elements of T_1 to T_2 with total multiplicities retained.

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Notation 1 Let $A = \{a_1,\ a_2,\ \cdots,\ a_k\}$ be a k-set then we will use the notation

$$\Delta A = [a_i - a_j :\ i \neq j,\ 1 \leq i,\ j \leq k].$$

We refer to [8] for the elementary theory of cyclotomy.

Let x be a primitive root of $GF(v)$ where $v = p^r = ef + 1$ is a prime power. Write $GF(v) \setminus \{0\}$. The *cyclotomic classes* C_i in $GF(v)$ are:

$$C_i = \{x^{es+i} :\ 0 \leq s \leq f-1\},\ 0 \leq i \leq e-1.$$

We note that C_i are pairwise disjoint and their union is $GF(v) \setminus \{0\}$.

If $n \equiv 2(mod\ 4)$, $v = \frac{1}{2}n$ and M, N are $v \times v$ commuting matrices, with elements ± 1, such that

$$MM^T + NN^T = (2v-2)I_v + 2J_v, \tag{3}$$

then the $n \times n$ matrix

$$R = \begin{bmatrix} M & N \\ -N^T & M^T \end{bmatrix}$$

has the maximum determinant (see [3, 5]) among all $n \times n \pm 1$ matrices. Such matrices are called D-optimal designs of order n.

Now form the two sets $A = \{a_1,\ a_2,\ \cdots,\ a_{k_1}\}$ and $B = \{b_1,\ b_2,\ \cdots,\ b_{k_2}\}$ where a_i, b_j denote the positions of the -1s in the first row of M, N respectively. If the matrices M, N are circulant, then they satisfy (3) if and only if, (see [2]), they are sds

$$2 - \{v;\ k_1,\ k_2;\ \lambda\} \quad \text{where} \quad \lambda = k_1 + \ k_2 - \frac{1}{4}(n-2) \quad \text{and} \quad k_1 \geq k_2 \geq 0$$

are found from

$$(v - 2k_1)^2 \ + \ (v - 2k_2)^2 = 2n - 2.$$

Hence the construction of the circulant matrices M and N satisfying (3) is equivalent to the construction of the corresponding sds. A similar result hold for group developed matrices and prime power orders.

2 Main Theorems

Koukouvinos, Kounias and Seberry [6] proved the following theorem:

Theorem 1 (see [6]) *There exist D-optimal designs of order* $n \equiv 2(mod\ 4)$ *where*

$$n = 2v = 2(q^2 + q + 1)$$

and q *is a prime power.*

A.L. Whiteman [11] proved the following theorem:

Theorem 2 (see [11]) *There exist D-optimal designs of order* $n \equiv 2(mod\ 4)$ *where*

$$n = 2v = 2(2q^2 + 2q + 1)$$

and q *is an odd prime power.*

The proof of Theorem 1 is based on supplementary difference sets whereas the proof of Theorem 2 is based on a remarkable construction of symmetric block designs due to A. E. Brouwer [1].

It is natural to ask if Theorem 2 can be established alternatively by means of a supplementary difference set construction.

The problem reduces to finding for each odd prime power q a pair of supplementary difference sets with parameters

$$v = 2q^2 + 2q + 1, \qquad k_1 = k_2 = q^2, \qquad \lambda = q(q-1).$$

The possibility that these supplementary difference sets also exist for q even is not precluded.

Theorem 3 *Let g be a generator of the cyclic group $GF(v) \setminus \{0\}$. Suppose*

(i) $v = 2q^2 + 2q + 1$ is a prime power,

(ii) A and B are $2 - \{v;\ q^2,\ q^2;\ \lambda\}$ sds such that $2q+1$ is a multiplier ie $B = (2q+1)A$, and $2q+1 \in C_i$,

(iii) A and B are unions of cyclotomic classes.

Then every $\alpha \in C_i$ or $\alpha \in C_i^{-1}$ is also a multiplier ie $B = \alpha A$.

Proof.

If q is even let

$$C_i = \{g^{4mj+i} :\ 0 \le j \le q\},\ 0 \le i \le 4m-1,\ \text{where } q = 2m.$$

We have $2q+1 \in C_m$ or $C_{3m} = C_m^{-1}$

$$A = \{0\} \cup C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_{2m-1}}$$

and

$$B = \{0\} \cup C_{i_1+m} \cup C_{i_2+m} \cup \cdots \cup C_{i_{2m-1}+m},$$

for all $\alpha \in C_m$, $\alpha A = \{0\} \cup \alpha C_{i_1} \cup \alpha C_{i_2} \cup \cdots \cup \alpha C_{i_{2m-1}}$, $\alpha = g^{4mj+m}$ for some j and $2q+1 = g^{4mj_0+m}$ for some j_0.

$$\begin{aligned} \alpha C_{i_1} &= \{g^{4mj+m} \cdot g^{4mk+i_1} : 0 \le k \le q\} \\ &= \{g^{4m(j+k)+i_1+m} : 0 \le k \le q\} \\ &= \{g^{4m\ell+i_1+m} : 0 \le \ell \le q\} \\ &= C_{i_1+m}. \end{aligned}$$

Also if $\alpha \in C_{3m}$ we proceed similarly with m replaced by $3m$.

If q is odd let

$$C_i = \{g^{4mj+i} :\ 0 \le j \le q-1\},\ 0 \le i \le 4m-1,\ \text{where } q = 2m-1$$

and the same proof follows. □

Theorem 4 *Suppose g is a generator of $GF(v)$ where $v = 4m+1$, m odd. If $(A,\ g^\alpha A)$ are 2-sds for some α then so are $(A,\ g^\beta A)$ for $\beta = 4m - \alpha,\ 2m + \alpha,\ 2m - \alpha$.*

Proof. Case $\beta = 4m - \alpha$:

$$\begin{aligned} \Delta A + \Delta g^{4m-\alpha} A &= \Delta A + g^{4m-\alpha} \Delta A \\ &= g^{4m} \Delta A + g^{4m-\alpha} \Delta A \\ &= g^{4m-\alpha}(g^{\alpha} \Delta A + \Delta A) \\ &= g^{4m-\alpha}(\Delta A + \Delta g^{\alpha} A). \end{aligned}$$

Case $\beta = 2m + \alpha$:

$$\Delta A + \Delta g^{2m+\alpha} A = \Delta A + \Delta g^{\alpha} A.$$

Case $\beta = 2m - \alpha$:

$$\begin{aligned} \Delta A + \Delta g^{2m-\alpha} A &= g^{2m} \Delta A + \Delta g^{2m-\alpha} A \\ &= g^{2m-\alpha}(\Delta A + g^{\alpha} \Delta A). \end{aligned}$$

□

3 Results

3.1 The case $q = 2$

Here $v = 13$ and we find $2 - \{13;\ 4,\ 4;\ 2\}$ sds by using 2 as a generator of $GF(13)$ and the cyclotomic classes

$$C_0 = \{1,\ 3,\ 9\}, \quad C_i = 2^i C_0, \quad i = 1,\ 2,\ 3.$$

Then

$$A = \{0\} \cup C_0, \qquad B = \{0\} \cup C_1 = 5A$$

are $2 - \{13;\ 4,\ 4;\ 2\}$sds, now $2q + 1 = 5 \in C_1$ and

$$\text{for all } a \in C_1 \cup C_3, \quad \text{we have} \quad B(\omega) = A(\omega^a),$$

where $\omega = e^{\frac{2\pi i}{v}}$ is a primitive vth root of unity.

3.2 The case $q = 3$

Here $v = 25$ (a prime power) and we find $2 - \{25;\ 9,\ 9;\ 6\}$ sds by using the solution x of $x^2 = x + 3$ as a generator of $GF(5^2)$ and the cyclotomic classes

$$C_0 = \{1,\ 2x + 1,\ 3x + 3\}, \quad C_i = x^i C_0, \quad i = 1,\ 2,\ \cdots,\ 7.$$

Then

$$A = C_0 \cup C_1 \cup C_4, \qquad B = x^2 A$$

are $2 - \{25;\ 9,\ 9;\ 6\}$ sds and every element in C_2 or C_6 is a multiplier so

$$\text{for all } a \in C_2 \cup C_6, \quad \text{we have} \quad B = aA.$$

3.3 The case $q = 4$

Here $v = 41$ and we find $2 - \{41;\ 16,\ 16;\ 12\}$ sds by using 6 as a generator of $GF(41)$ and the cyclotomic classes

$$C_0 = \{1,\ 10,\ 16,\ 18,\ 37\}, \quad C_i = 6^i C_0, \quad i = 1,\ 2,\ \cdots,\ 7.$$

Then

$$A = \{0\} \cup C_1 \cup C_5 \cup C_6, \qquad B = 6^6 A$$

are $2 - \{41;\ 16,\ 16;\ 12\}$ sds and every element in C_2 or C_6 is a multiplier. We note $2q + 1 = 9 \in C_6$ and

$$\text{for all } a \in C_2 \cup C_6, \quad \text{we have} \quad B(\omega) = A(\omega^a),$$

where $\omega = e^{\frac{2\pi i}{v}}$ is a primitive vth root of unity.

3.4 The case $q = 5$

Here $v = 61$ and we find $2 - \{61;\ 25,\ 25;\ 20\}$ sds by using 2 as a generator of $GF(61)$ and the cyclotomic classes

$$C_0 = \{1,\ 9,\ 20,\ 34,\ 58\}, \quad C_i = 2^i C_0, \quad i = 1,\ 2,\ \cdots,\ 11.$$

Then

$$A = C_0 \cup C_2 \cup C_3 \cup C_5 \cup C_6, \qquad B = 2^3 A$$

are $2 - \{61;\ 25,\ 25;\ 20\}$ sds. We note $2q + 1 = 11 \in C_3$ and

$$\text{for all } a \in C_3 \cup C_9, \quad \text{we have} \quad B(\omega) = A(\omega^a),$$

where $\omega = e^{\frac{2\pi i}{v}}$ is a primitive vth root of unity.

3.5 The case $q = 7$

Here $v = 113$ and we find $2 - \{113;\ 49,\ 49;\ 42\}$ sds by using 3 as a generator of $GF(113)$ and the cyclotomic classes

$$C_0 = \{1,\ 16,\ 28,\ 30,\ 49,\ 100,\ 109\}, \quad C_i = 3^i C_0, \quad i = 1,\ 2,\ \cdots,\ 15.$$

Then

$$A = C_0 \cup C_1 \cup C_3 \cup C_5 \cup C_7 \cup C_8 \cup C_{12}, \qquad B = 3^4 A$$

are $2 - \{113;\ 49,\ 49;\ 42\}$ sds. We note $2q + 1 = 15 \in C_4$ and

$$\text{for all } a \in C_4 \cup C_{12}, \quad \text{we have} \quad B(\omega) = A(\omega^a),$$

where $\omega = e^{\frac{2\pi i}{v}}$ is a primitive vth root of unity.

3.6 The case $q = 9$

Here $v = 181$ and we find $2 - \{181;\ 81,\ 81;\ 72\}$ sds by using 2 as a generator of $GF(181)$ and the cyclotomic classes

$$C_0 = \{1,\ 39,\ 43,\ 48,\ 62,\ 65,\ 73,\ 80,\ 132\}, \quad C_i = 2^i C_0, \quad i = 1,\ 2,\ \cdots,\ 19.$$

Then

$$A = C_0 \cup C_1 \cup C_7 \cup C_{10} \cup C_{11} \cup C_{13} \cup C_{14} \cup C_{16} \cup C_{19}, \qquad B = 2^5 A$$

are $2 - \{181;\ 81,\ 81;\ 72\}$ sds. We note $2q + 1 = 19 \in C_{15}$ and

$$\text{for all } a \in C_5 \cup C_{15}, \quad \text{we have} \quad B(\omega) = A(\omega^a),$$

where $\omega = e^{\frac{2\pi i}{v}}$ is a primitive vth root of unity.

Conjecture 1 *For each prime power q there exists a pair of supplementary difference sets A and B with parameters*

$$v = 2q^2 + 2q + 1, \qquad k_1 = k_2 = q^2, \qquad \lambda = q(q-1),$$

where v is a prime power and the relation

$$B(\omega) = A(\omega^{2q+1})$$

holds where $\omega = e^{\frac{2\pi i}{v}}$ is a primitive vth root of unity.

4 Some Remarks on Results of Dokovic

We note that if $C_a \cup C_b \cup \cdots \cup C_c$ and $C_d \cup C_e \cup \cdots \cup C_f$ are two sds then $C_{a+j} \cup C_{b+j} \cup \cdots \cup C_{c+j}$ and $C_{d+j} \cup C_{e+j} \cup \cdots \cup C_{f+j}$ are also two sds.

4.1 The case $q = 4$

The primitive element 6 was used to generate $GF(41)$. Then the cyclotomic classes

$$C_i = \{6^{8j+i} :\ 0 \le j \le 5\},\ 0 \le i \le 8$$

were formed.

Then using Dokovic [4, p374], where he uses the α_k notation for the cosets and we use the C_i notation for our cyclotomic classes, we have the correspondence

α_k	0	1	2	3	4	5	6	7
C_i	0	4	2	6	7	3	1	5

The sds given in [4, p374] were transformed using the mapping $\alpha_{k_x} \to C_{i_x}$ and then multiplied as indicated.

case	map	$2^x P$	$2^x Q$
(a)	6^0	$A = C_0 \cup C_2 \cup C_3 \cup C_4 \cup C_7$	$6^6 A = C_0 \cup C_1 \cup C_2 \cup C_5 \cup C_6$
(b)	6^6	$6^3 A = C_2 \cup C_3 \cup C_5 \cup C_6 \cup C_7$	$B = C_0 \cup C_1 \cup C_2 \cup C_4 \cup C_5$
(c)	6^0	$B = C_0 \cup C_1 \cup C_2 \cup C_4 \cup C_5$	$6^1 B = C_1 \cup C_2 \cup C_3 \cup C_5 \cup C_6$
(d)	6^0	$C = C_0 \cup C_2 \cup C_3 \cup C_4 \cup C_6$	$D = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_6$

For case (a) we have $B(\omega) = A(\omega^9)$ where $2q+1 = 9 \in C_6$. We note that $-6^4A = B$. So case (a) is $(A,\ 6^6A)$, case (b) is $(6^3A, B)$, and case (c) is $(B, 6^1B)$. We showed in our Theorem 3 that any element of $C_2 \cup C_6$ is a multiplier for A. By Theorem 4 if g^α is a multiplier then so are $g^{40-\alpha}$, $g^{20+\alpha}$ and $g^{20-\alpha}$. Using our notation these are inequivalent $2 - \{41;\ 25,\ 25;\ 30\}$.

The inequivalent $2 - \{41;\ 16,\ 16;\ 12\}$ are given below:

case	map	2^xP	2^xQ
(a)	6^0	$A' = \{0\} \cup C_1 \cup C_5 \cup C_6$	$6^6A' = \{0\} \cup C_3 \cup C_4 \cup C_7$
(b)	6^6	$6^3A' = \{0\} \cup C_0 \cup C_1 \cup C_4$	$B' = \{0\} \cup C_3 \cup C_6 \cup C_7$
(c)	6^0	$B' = \{0\} \cup C_3 \cup C_6 \cup C_7$	$6^1B' = \{0\} \cup C_0 \cup C_4 \cup C_7$
(d)	6^0	$C' = \{0\} \cup C_1 \cup C_5 \cup C_7$	$D' = \{0\} \cup C_4 \cup C_5 \cup C_7$

4.2 The case $q = 5$

The primitive element 2 was used to generate $GF(61)$. Then the cyclotomic classes

$$C_i = \{2^{12j+i} :\ 0 \le j \le 5\},\ 0 \le i \le 11$$

were formed.

Then using Dokovic [4, p375], where he uses the α_k notation for the cosets and we use the C_i notation for our cyclotomic classes, we have the correspondence

α_k	0	1	2	3	4	5	6	7	8	9	10	11
C_i	0	6	1	7	2	8	10	4	3	9	11	5

The sds given in [4, p375] were transformed using the mapping $\alpha_{k_x} \to C_{i_x}$ and then multiplied as indicated.

case	map	2^xP	2^xQ
(a)	2^7	$A = C_0 \cup C_4 \cup C_7 \cup C_9 \cup C_{10}$	$B = C_0 \cup C_1 \cup C_2 \cup C_6 \cup C_{10}$
(h)	2^3	$2^6A = C_1 \cup C_3 \cup C_4 \cup C_6 \cup C_{10}$	$B = C_0 \cup C_1 \cup C_2 \cup C_6 \cup C_{10}$
(j)	2^7	$B = C_0 \cup C_1 \cup C_2 \cup C_6 \cup C_{10}$	$2^6A = C_1 \cup C_3 \cup C_4 \cup C_6 \cup C_{10}$
(b)	2^1	$E = C_1 \cup C_3 \cup C_6 \cup C_7 \cup C_{11}$	$D = C_0 \cup C_1 \cup C_4 \cup C_6 \cup C_8$
(c)	2^{10}	$D = C_0 \cup C_1 \cup C_4 \cup C_6 \cup C_8$	$2^6E = C_0 \cup C_1 \cup C_5 \cup C_7 \cup C_9$
(e)	2^2	$2^3E = C_2 \cup C_4 \cup C_6 \cup C_9 \cup C_{10}$	$D = C_0 \cup C_1 \cup C_4 \cup C_6 \cup C_8$
(i)	2^9	$2^9E = C_0 \cup C_3 \cup C_4 \cup C_8 \cup C_{10}$	$D = C_0 \cup C_1 \cup C_4 \cup C_6 \cup C_8$
(d)	2^4	$F = C_0 \cup C_1 \cup C_3 \cup C_4 \cup C_{10}$	$G = C_0 \cup C_5 \cup C_8 \cup C_{10} \cup C_{11}$
(g)	2^7	$2^9F = C_0 \cup C_1 \cup C_7 \cup C_9 \cup C_{10}$	$F = C_0 \cup C_1 \cup C_3 \cup C_4 \cup C_{10}$
(f)	2^3	$H = C_1 \cup C_3 \cup C_6 \cup C_7 \cup C_{10}$	$I = C_0 \cup C_1 \cup C_2 \cup C_5 \cup C_6$

One of these is $11 \cdot P = Q$. We will show that $\Delta(C_i) = \Delta(-C_i) = \Delta(2^{12j}C_i) = \Delta(2^{12j+6}C_i)$. This means that since $2^9 \in C_9$ any element of C_9 or C_3 can act as a multiplier and, noting $11 \in C_3$, we have for case (g) that $B(\omega) = A(\omega^{11})$ and $B(\omega) = A(-\omega^{11})$.

We note that

$$\Delta(A) = \{x - y : x \in A,\ y \in A\}$$

Suppose $d = x - y$ has δ_{x-y} solutions in $\Delta(A)$ then $-d = y - x$ has the same number of solutions δ_{x-y}. This means that in $\Delta(-A)$ we have $-d = -x - (-y)$ and $d = -y - (-x)$ also have δ_{x-y} solutions. Hence $\Delta(A) = \Delta(-A) = 2^{60}\Delta(A) = a\Delta(A)$ for every $a \in C_0$ or C_6.

This means cases (a), (h) and (j) are equivalent. Cases (b) and (c) are equivalent; cases (e) and (i) are equivalent; and the cases (d), (g) and (f) are inequivalent to all of these. Hence there are six equivalence classes when group multipliers are taken into account.

case	2^xP	2^xQ
(a)	$A = C_0 \cup C_4 \cup C_7 \cup C_9 \cup C_{10}$	$B = C_0 \cup C_1 \cup C_2 \cup C_6 \cup C_{10}$
(b)	$E = C_1 \cup C_3 \cup C_6 \cup C_7 \cup C_{11}$	$D = C_0 \cup C_1 \cup C_4 \cup C_6 \cup C_8$
(e)	$D = C_0 \cup C_1 \cup C_4 \cup C_6 \cup C_8$	$2^3E = C_2 \cup C_4 \cup C_6 \cup C_9 \cup C_{10}$
(d)	$F = C_0 \cup C_1 \cup C_3 \cup C_4 \cup C_{10}$	$G = C_0 \cup C_5 \cup C_8 \cup C_{10} \cup C_{11}$
(g)	$F = C_0 \cup C_1 \cup C_3 \cup C_4 \cup C_{10}$	$2^3F = C_1 \cup C_3 \cup C_4 \cup C_6 \cup C_7$
(f)	$H = C_1 \cup C_3 \cup C_6 \cup C_7 \cup C_{10}$	$I = C_0 \cup C_1 \cup C_2 \cup C_5 \cup C_6$

5 More Results in the Cyclic Groups

5.1 The case $q = 3$

We have also found the following example which is inequivalent to those mentioned above:

$$A = \{0,\ 1,\ 2,\ 6,\ 8,\ 10,\ 11,\ 14,\ 15\}, \quad B = 7A.$$

These are $2 - \{25;\ 9,\ 9;\ 6\}$ sds.

5.2 The case $q = 4$

We have also found the following example which is inequivalent to those mentioned above:

$$A = \{0,\ 1,\ 7,\ 14,\ 17,\ 20,\ 21,\ 22,\ 25,\ 26,\ 27,\ 29,\ 31,\ 37,\ 39,\ 40\}, \quad B = 9A.$$

These are $2 - \{41;\ 16,\ 16;\ 12\}$ sds.

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