CRITICAL SETS IN ROOM SQUARES

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Abstract

In this paper we introduce critical sets in Room squares. We give the cardinality of the small s-critical sets (*ss-cs*) and large s-critical sets (*ls-cs*) for different classes of standardized Room squares of side 7, 9 and 11. We also describe algorithms to compute *ss-cs* and *ls-cs* and conjecture lower and upper bounds for *ss-cs* and *ls-cs*. We are modifying these to obtain proofs for the minimum and maximum size of *s-cs* and *l-cs*.

1 Introduction

A *Room square* R of side r is an $r \times r$ array each of whose cells may either be empty or contain an unordered pair of objects $0, 1, 2, ..., r$, subject to the following conditions :

(i) each of the objects $0, 1, 2, ..., r$ occurs precisely once in each row of R and precisely once in each column of R, and

(ii) every possible unordered pair of objects occurs precisely once in the whole array.

Theorem 1 (Mullin and Wallis [11]) *There exists a Room square of every odd integer side r greater than or equal to 7.*

Theorem 2 (Dinitz and Stinson [6]) *A Room square of side r is equivalent to a pair of orthogonal one-factorizations of a complete graph.*

We denote $N_r = 0, 1, 2, ...r$. A Room square of side r based on N_r is *standardized* if the ith diagonal cell, cell(i, i), contains $\{0, i\}$ for $1 \leq i \leq r$. Throughout this paper, we will be working on standardized Room squares only.

Theorem 3 (Wallis [15]) *If there is a Room square of side r, then there is a standardized Room square of side r.*

A *skew* Room square R [14] is a standardized Room square with the property that when $i \neq j$, either the (i, j) or the (j, i) cell of R is occupied, but not both.

Theorem 4 (Dinitz and Stinson [6]) *There is a skew Room square of side r, if and only if r is odd and $r \neq 3$ or 5.*

Two Room squares are said to be *isomorphic* [15] if one can be obtained from the other by permuting the rows and columns and relabeling the elements. Two Room squares R and S are *equivalent* if R is isomorphic to S or to the transpose of S. All inequivalent Room squares of size 7 and 9 have been found from one-factorizations of the complete graph. There are exactly 6 inequivalent Room squares of side 7, see [8, 9, 15] and 257,630 Room squares for side 9, see [1, 6, 7]. No exact number is known for higher order Room squares. Although there are :

526,915,620 non-isomorphic one-factorizations of order 11 ($K12$)
9.876×10^{28} distinct one-factorizations of order 13 ($K14$)
1.148×10^{44} distinct one-factorizations of order 15 ($K16$)
1.520×10^{63} distinct one-factorizations of order 17 ($K18$)
and the number of possible Room squares for each of these orders should be even larger, see [4] for further details.

2 Critical Sets in Room Squares

A *critical set* $Q = [Q_1, Q_2, Q_3, ..., Q_c]$, $|Q| = c$, in a Room square R of side r, is a set of quadruples $Q_a = [i, j, k, l]$ such that if any Q_a is removed from the set it can no longer be uniquely completed. In Q_a, (i, j) shows the position of the pair (k, l) in the square. That is, Q provides minimal information from which R can be reconstructed uniquely.

Q^* is a *minimal/smallest critical set (scs)* of a Room square R if $|Q^*| = c$ is minimum for all critical sets Q of squares of order r whereas Q^* is a *largest critical set (lcs)* of a Room square R if $|Q^*| = c$ is the largest for all critical sets Q of squares of order r. In particular, in this paper, we define *large s-critical set (ls-cs)* and *small s-critical set (ss-cs)* as critical sets of a standardized (i.e. known diagonal) Room square.

Example

Standardized Room square of side 7:

01	-	45	67	-	-	23
57	02	-	-	-	13	46
-	56	03	12	-	47	-
-	37	-	04	26	-	15
36	14	27	-	05	-	-
24	-	-	35	17	06	-
-	-	16	-	34	25	07

Large s-critical set *(ls-cs)* of this square is:

01	-	45	*	-	-	23
*	02	-	-	-	*	*
-	*	03	12	-	*	-
-	*	-	04	*	-	*
*	*	27	-	05	-	-
*	-	-	*	17	06	-
-	-	*	-	*	*	07

where "$*$" shows the unknown pair positions and "$-$" shows empty positions in the square.

So the large s-critical set for the above Room square is: $Q = \{(1,3;45)$, $(1,7;23)$, $(3,4;12)$, $(5,3;27)$, $(6,5;17)\}$. That is, Q is the only Room square of side 7 with a pair 45 at position (1,3), pair 23 at position (1,7), pair 12 at position (3,4), pair 27 at position (5,3) and pair 17 at position (6,5).

2.1 Bounds for Critical Sets

We have computed large *(ls-cs)* and small *(ss-cs)* s-critical sets for all inequivalence classes of standardized Room squares of side 7 and some of the squares for side 9 and 11. We summarize our results in the table 1 below.

Table 1

Order of Room square	Class Representative	*Cardinality* ls-cs	ss-cs
7	R11	6	3
	R14	4	3
	R15	5	3
	R16	5	3
	R44	5	3
	R45	4	3
7 Skew	R11	6	3
	R15$\tilde{6}$	4	3
	R45$\tilde{5}$	6	3
9	R1	11	7
	R2	12	8
	R3	11	9
	R4	11	7
	R5	10	7
	R6	11	8
9 Skew	R1	11	7
11	R1	18	15
	R2	18	15
	R3	17	13
	R4	18	13

2.2 Algorithms for constructing *ss-cs* and *ls-cs*

2.2.1 Algorithm for *ss-cs*

Suppose we have a standardized Room square R of side r, take an empty square Q corresponding to R.

1) for each row of R, mask off first non-zero pair.

2) for each column of R, mask off last non-zero pair.

3) for each row from bottom, process following steps:

   4) for each column of a row, process following steps:

      5) if a pair exists at current column and row, then mask off that pair.
      6) create set of known pairs.
      7) call *check_critical_set* routine.
      8) if known pair set is complete, then continue else put that pair back in R and also move that pair in Q.

9) Q is the required small s-critical set.

2.2.2 Algorithm for *ls-cs*

1) Suppose we have a standardized Room square R of side r, take empty squares S and Q corresponding to R.

2) take $[\frac{1}{2}(r-1)-1]$ pairs from the first row of R, move these pairs to Q, construct at least first row of S by calling *check_critical_set* routine.

3) take another pair from R so that more unknown pairs of S can be found, move that pair to Q too and

   4) call *check_critical_set* routine to find more unknown pairs of S.

5) repeat step 3 and 4 until S is complete and Q is the required large s-critical set.

2.2.3 Algorithm for *check_critical_set* routine

1) initialize known and unknown pairs.

2) for each row of R from top, process steps 3-8:

   3) check for any known pairs, if none, process next row and go to step 2.
   4) form known numbers.
   5) for each unknown pair position, construct known numbers for its column.
   6) take union of row numbers with each of column numbers (call it as set Z)
   7) take intersection for each element of Z with all numbers from 1 to r.
   8) if after taking intersection only two numbers are left, then add it to the known set and go to step 2 otherwise make all pairs of the remaining numbers and discard the pairs which are in the known set, after discarding if one pair is left, add it to known set and go to step 2 otherwise go to step 5.

2.2.4 Time and Space Complexity of Algorithms

Algorithm	Time	Space
ss-cs	$O(n^6)$	$O(n^2)$
ls-cs	$O(n^7)$	$O(n^2)$
chk_crit_set routine	$O(n^4)$	$O(n^2)$

We are working to improve the time complexity of above algorithms.

2.3 Conjecture

On the basis of our results, we conjecture that for a Room square R of side r, lower and upper bounds for small and large s-critical sets are :

(i) $ss\text{-}cs(r) \leq 1 + \frac{1}{4}(r-5)(r-3)$

(ii) $ls\text{-}cs(r) \geq 4 + \frac{1}{8}(r+15)(r-7)$

Appendix

The following are examples of small and large s-critical sets of standardized Room squares listed in table 1. In s-critical sets, '**' represent unknown pair positions and '-' show the empty positions in the squares.

Example 1 : Standardized Room Square of side 7

01	-	45	67	-	-	23
57	02	-	-	-	13	46
-	56	03	12	-	47	-
-	37	-	04	26	-	15
36	14	27	-	05	-	-
24	-	-	35	17	06	-
-	-	16	-	34	25	07

Three s-critical sets for this square are :

0,1	–	**	6,7	–	–	2,3
**	0,2	–	–	–	1,3	4,6
–	**	0,3	**	–	4,7	–
–	**	–	0,4	**	–	**
**	**	**	–	0,5	–	–
**	–	–	**	**	0,6	–
–	–	**	–	**	**	0,7

0,1	–	**	**	–	–	**
**	0,2	–	–	–	**	**
–	**	0,3	1,2	–	**	–
–	3,7	–	0,4	**	–	**
**	**	**	–	0,5	–	–
**	–	–	**	1,7	0,6	–
–	–	**	–	**	**	0,7

0,1	–	**	6,7	–	–	**
**	0,2	–	–	–	1,3	4,6
–	**	0,3	1,2	–	**	–
–	**	–	0,4	**	–	**
**	**	2,7	–	0,5	–	–
**	–	–	**	**	0,6	–
–	–	**	–	**	**	0,7

Example 2 : Skew Room square of side 7

0,1	3,7	–	2,5	4,6	–	–
–	0,2	1,5	3,6	–	4,7	–
2,6	–	0,3	1,7	–	–	4,5
–	–	–	0,4	2,7	3,5	1,6
–	1,4	6,7	–	0,5	–	2,3
5,7	–	2,4	–	1,3	0,6	–
3,4	5,6	–	–	–	1,2	0,7

Three s-critical sets for this square are :

0,1	**	–	2,5	4,6	–	–
–	0,2	**	**	–	**	–
**	–	0,3	**	–	–	**
–	–	–	0,4	**	**	1,6
–	**	**	–	0,5	–	**
**	–	**	–	**	0,6	–
**	**	–	–	–	**	0,7

0,1	**	–	2,5	**	–	–
–	0,2	1,5	3,6	–	**	–
**	–	0,3	**	–	–	**
–	–	–	0,4	**	**	**
–	**	6,7	–	0,5	–	**
**	–	**	–	1,3	0,6	–
**	**	–	–	–	**	0,7

0,1	**	–	**	**	–	–
–	0,2	**	**	–	**	–
**	–	0,3	**	–	–	**
–	–	–	0,4	**	**	**
–	**	**	–	0,5	–	**
**	–	2,4	–	**	0,6	–
3,4	5,6	–	–	–	**	0,7

Example 3 : Standardized Room Square of side 9

0,1	–	4,9	3,7	2,8	–	5,6	–	–
8,9	0,2	–	–	–	5,7	3,4	–	1,6
–	5,8	0,3	–	6,9	2,4	–	1,7	–
–	3,6	7,8	0,4	–	1,9	–	2,5	–
–	7,9	–	1,2	0,5	3,8	–	4,6	–
4,5	–	–	–	–	0,6	1,8	3,9	2,7
–	–	2,6	5,9	1,3	–	0,7	–	4,8
6,7	1,4	–	–	–	–	2,9	0,8	3,5
2,3	–	1,5	6,8	4,7	–	–	–	0,9

Two s-critical sets for this square are :

0,1	–	**	3,7	2,8	–	5,6	–	–
**	0,2	–	–	–	5,7	3,4	–	**
–	**	0,3	–	**	2,4	–	1,7	–
–	**	**	0,4	–	1,9	–	2,5	–
–	**	–	**	0,5	**	–	**	–
**	–	–	–	–	0,6	**	**	**
–	–	**	**	**	–	0,7	–	**
**	**	–	–	–	–	**	0,8	**
**	–	**	**	**	–	–	–	0,9

0,1	–	**	**	**	–	**	–	–
**	0,2	–	–	–	**	**	–	**
–	5,8	0,3	–	6,9	2,4	–	**	–
–	**	**	0,4	–	1,9	–	**	–
–	7,9	–	**	0,5	**	–	**	–
**	–	–	–	–	0,6	**	**	**
–	–	**	5,9	1,3	–	0,7	–	**
**	**	–	–	–	–	**	0,8	**
**	–	**	**	**	–	–	–	0,9

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