On the Smith Normal Form of Weighing Matrices

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Abstract

The Smith normal forms (SNF) of weighing matrices are studied.

We show that for all orders $n \geq 49$ the *full spectrum* of Smith normal forms (SNF) exists for weighing matrices $W(n,9)$ i.e. there exists a $W(n,9)$ with SNF $1^{\frac{1}{2}(n-s)}3^s 9^{\frac{1}{2}(n-s)}$, for s in a set, which is described, of consecutive even or odd integers.

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1 Introduction

The general question which gave rise to the present paper is the question of isomorphism of weighing matrices. Theoretically it is enough to check $(n!)^2$ cases to find if two $n \times n$ given incidence matrices are permutation equivalent. In practice it is not feasible in any interesting case, even despite the fact that the number of comparisons can be reduced to $n!$ as was shown by M.Newman [**?**]. The Smith normal form may be used for a negative answer : if two incidence matrices have different Smith normal forms then they are not permutation equivalent. It is thus interesting to learn more about the Smith normal form for weighing matrices. First we introduce some notation and summarize results on the Smith normal form and weighing matrices. Let $\mathbf{R_n}$ be the ring of $n \times n$ matrices with entries from $\mathbf{Z}$, the ring of integers. The unit (*or* unimodular) elements of $\mathbf{R_n}$ are those with determinant ± 1.

Definition 1 The matrices A, B of $\mathbf{R_n}$ are equivalent (or $\mathbf{Z}$- equivalent, written $B \sim A$) if $B = PAQ$ for some unimodular matrices P, Q of $\mathbf{R_n}$.

The following Theorem is due to Smith [**?**] and has been reworded from the theorems and proofs in MacDuffee [**?**, p41] and Marcus and Minc [**?**, p44].

Theorem 1 *If $A = (a_{ij})$ is any integer (or with elements from any Euclidean domain) matrix of order n and rank r, then there is a unique integer (or with elements from any Euclidean domain) matrix*

$$D = \mathrm{diag}(a_1,\ a_2,\ \cdots,\ a_r,\ 0,\ \cdots,\ 0),$$

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such that $A \sim D$ *and* $a_i \mid a_{i+1}$ *where the* a_i *are non-negative. The greatest common divisor of the* $i \times i$ *subdeterminants of* A *is*

$$a_1 a_2 \cdots a_i.$$

If $A \sim E$ *where*

$$E = \mathrm{diag}(a_1,\ a_2,\ \cdots\ ,\ a_i) \oplus\ F$$

then a_{i+1} *is the greatest common divisor of the non-zero elements of* F. □

Definition 2 The a_i of theorem **??** are called the *invariants* of A and the diagonal matrix D is called the *Smith normal form (SNF).*

In this paper we will study the Smith normal forms of weighing matrices.

An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words

$$AA^T = (s_1 x_1^2 + \ldots + s_u x_u^2) I_n$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d$, $0 \le d < 4$, then $\rho(n) = 8c + 2^d$.

A *weighing matrix* $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n\,;k)$. The number k is called the *weight* of W.

2 SNF via Kronecker Products

Lemma 1 *Suppose the SNF of a matrix* $A = W(n,k)$ *is* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *where* $\lambda_i \mid \lambda_{i+1}$. *Then the SNF of one* $W(mn, kl)$ *is derived from*

(i) where $l = m = 2$ *and* $W(2n, 2k) = \begin{pmatrix} A & A \\ A & -A \end{pmatrix}$,

$$S = (\lambda_1, \lambda_2, \ldots, \lambda_n, 2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n);$$

(ii) where $l = 3$, $m = 4$ *and*

$$W(4n, 3l) = \begin{pmatrix} 0 & A & A & A \\ A & 0 & A & -A \\ A & -A & 0 & A \\ A & A & -A & 0 \end{pmatrix},$$

$$S = (\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n, 3\lambda_1, \ldots, 3\lambda_n, 3\lambda_1, \ldots, 3\lambda_n);$$

(iii) where $l = 4$, $m = 4$ *and*

$$W(4n, 4l) = \begin{pmatrix} A & A & A & A \\ A & A & -A & -A \\ A & -A & A & -A \\ A & -A & -A & A \end{pmatrix},$$

$$S = (\lambda_1, \ldots, \lambda_n, 2\lambda_1, \ldots, 2\lambda_n, 2\lambda_1, \ldots, 2\lambda_n, 4\lambda_1, \ldots, 4\lambda_n),$$

by using the operations

(a) order the elements of S in ascending order and write as $(\mu_1, \ldots, \mu_s)$;

(b) if $\mu_i \not| \mu_{i+1}$, $g \mid \mu_i$ and $g \mid \mu_{i+1}$, $g \neq 1$, replace μ_i and μ_{i+1} by μ_i/g and $g\mu_{i+1}$. Repeat steps (a) and (b) until the μ_i are in ascending order and $\mu_i \mid \mu_{i+1}$ for each $i = 1, \ldots, s-1$. □

Example 1 SNF $(W(6,4)) = (1^2\ 2^2\ 4^2)$, so there are $W(12,8)$ and $W(24,16)$ with SNFs $(1^2\ 2^4\ 4^4\ 8^2)$ and $(1^2\ 2^6\ 4^8\ 8^6\ 16^2)$ respectively however the SNF of the $W(24,12)$ obtained this way will be $(1^4\ 2^8\ 6^8\ 12^4)$.

Remark 1 The previous Lemma can be written more generally to encompass the SNF of the Kronecker product of any two matrices. We are concerned only with small weighing matrices.

3 Numerical Results for SNF of $W(n,9)$

The negacyclic $W(20,9)$ with first row $1100-1--0-000-000010$ has SNF, $1^6 3^8 9^6$.

We use the sequences $\{x_1, x_2, -x_3, 0\}$, $\{x_1, -x_2, 0, x_4\}$, $\{x_1, 0, x_3, -x_4\}$, $\{x_2, x_3, x_4, 0\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16;3,3,3,3)$. With the variables chosen as $x_1 = x_3 = x_4 = 1, x_2 = 0$, and as $x_1 = 0, x_2 = x_3 = x_4 = 1$ we obtain $W(16,9)$ with SNF, $1^6 3^4 9^6$.

Using these two results, those from Chan, Rodger and Seberry [**?**, Appendix A] and Geramita and Seberry [**?**] we have the results in Table 2 for $10 \leq n \leq 20$.

By processing Ohmori's [**?**, p.76] results through our computer program (after correcting a few typographical errors) we found that:

SNF	Type
$1^3 3^7 9^3$	W_1^*
$1^4 3^5 9^4$	W_4^*
$1^5 3^3 9^5$	W_2^*, W_3^*, W_7^*
$1^6 3^1 9^6$	W_5^*, W_6^*, W_8^*

Table 1: SNF for $W(13,9)$

Hence there exist $W(13,9)$ with SNF $1^{\frac{1}{2}(13-b)} 3^b 9^{\frac{1}{2}(13-b)}$ for $b = 1, 3, 5$ and 7.

We now consider how to combine known results to make larger results. By MacDuffee [**?**] and Marcus and Minc [**?**] we know that if the SNF of A is $1^a k^b (lk)^a$ and the SNF of B is $1^\alpha k^\beta (lk)^\alpha$ then the SNF of $A \oplus B$ is $1^{a+\alpha} k^{b+\beta} (lk)^{a+\alpha}$.

Example 2 The SNF, $A_b = W(12,9)$ is $1^a 3^b 9^a$ for $b = 0, 2, 4$. Hence all the combinations, $A_{b_1} \oplus A_{b_2}$, $b_1, b_2 = 0, 2, 4$ give $W(24,9)$ with SNF $1^x 3^y 9^x$ for $x = 0, 2, 4, 6, 8$. Indeed we can ensure that there is a $W(12s,9)$ with SNF, $1^{6s-t} 3^{2t} 9^{6s-t}$, for every $2t = 0, 2, \ldots, 4s$. This shows $W(60,9)$ exist with the required SNF for every $2t = 0, 2, \ldots, 20$. However this is not the best possible.

Definition 3 (Full Spectrum) We say the *full spectrum* SNF exists for $W(n,9)$ if for a given n, there exists a $W(n,9)$ with SNF $1^{\frac{1}{2}(n-s)} 3^s 9^{\frac{1}{2}(n-s)}$, where for

(i) $n \equiv 0(mod\,4)$, s takes the values $s = 0, 2, 4, \ldots, \sigma$,

(ii) $n \equiv 2(mod\,4)$, s takes the values $s = 2, 4, 6, \ldots, \sigma$,

(iii) $n \equiv 1, 3(mod\,4)$, s takes the values $s = 1, 3, 5, \ldots, \sigma$.

Lemma 2 *If the full spectrum SNF exists for $W(n,9)$ and the full spectrum SNF exists for $W(m,9)$ then the full spectrum SNF exists for $W(n+m,9)$,*

(a) where $n \equiv 0(mod\,4)$ and m is any value;

(b) except for $s = 0$, when $n \equiv 2(mod\,4)$ and $m \equiv 0(mod\,4)$;

(c) except for $s = 1$, when $n \equiv 2(mod\,4)$ and $m \equiv 1, 3(mod\,4)$;

(d) except for $s = 0, 2$, when $n \equiv 2(mod\,4)$ and $m \equiv 2(mod\,4)$;

(e) except for $s = 0$ when $n \equiv 1(mod\,4)$ and $m \equiv 3(mod\,4)$.

In general if the upper bounds for the spectrum of A and B are a, b respectively the upper bound for $A \oplus B$ will be $a + b$. □

Definition 4 Full Spectrum Sets The full spectrum of $W(n,9)$ is given by

(i) $s = 0,\ 2, \cdots,\ \sigma$ when $n \equiv 0(mod\,4)$,

(ii) $s = 2,\ 4, \cdots,\ \sigma$ when $n \equiv 2(mod\,4)$,

(iii) $s = 1,\ 3, \cdots,\ \sigma$ when $n \equiv 1, 3(mod\,4)$,

where $\sigma = 7k,\ 7k-5,\ 7k-2,\ 7k+1,\ 7k-4,\ 7k-1,\ 7k+2,\ 7k+1,\ 7k,\ 7k+3,\ 7k+2,\ 7k+1,\ 7k+4$, according as $n = 13k,\ 13k+1,\ 13k+2,\ 13k+3,\ 13k+4,\ 13k+5,\ 13k+6,\ 13k+7,\ 13k+8,\ 13k+9,\ 13k+10,\ 13k+11,\ 13k+12$, i.e. there exists a $W(n,9)$ with SNF $1^{\frac{1}{2}(n-s)}3^s9^{\frac{1}{2}(n-s)}$, for these values for s.

Theorem 2 (Full Spectrum Theorem) *The full spectrum of $W(n,9)$ exists for all $n \geq 49$ except*

(i) for $k = 0$, there is no $W(11,9)$;

(ii) for $k = 1$, the cases $n = 15,\ 16,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22$ and 23 are unsettled;

(iii) for $k = 2$, the cases $n = 30,\ 31,\ 32,\ 34$ and 35 are unsettled;

(iv) for $k = 3$, the cases $n = 47$ and 48 are unsettled.

Proof. We note from Table 2 that orders

$$13,\ 14,\ 28,\ 29,\ 43,\ 44,\ 45,\ 33,\ 60,\ 61,\ 23,\ 24,\ 25$$

satisfy the conditions of the enunciation. These numbers are all in different equivalence classes modulo 13. Hence by adding suitable numbers of copies of W_{13} which have SNF either $1^6 3^1 9^6$, $1^5 3^3 9^5$,$1^4 3^5 9^4$ or $1^3 3^7 9^3$, as required we get new $W(n+13,9)$ with SNFs also satisfying the theorem. □

We conjecture the theorem is true for all $n \geq 22$.

label	order	s range	method	number
W_{10}*	10	2	construction	1
W_{11}	-		doesn't exist	
W_{12}*	12	$0, 2, 4$	construction	3
W_{13}*	13	$1, 3, 5, 7$	construction	4
W_{14}*	14	2	construction	≥ 1
W_{15}	15	1		≥ 1
W_{16}	16	$0, 4, 8$	construction	≥ 3
W_{17}	-			
W_{18}	18	2	construction	≥ 1
W_{19}	19	1		≥ 1
W_{20}	20	$0, 4, 8$	construction	≥ 3
W_{21}	-			
W_{22}	22	$2, 4, 6$	$W_{10} \oplus W_{12}$	≥ 3
W_{23}	23	$3, 5, 7, 9$	$W_{10} \oplus W_{13}$	≥ 4
W_{24}*	24	$0, 2, 4, 6, 8$	$W_{12} \oplus W_{12}$	5
W_{25}*	25	$1, 3, \ldots, 11$	$W_{12} \oplus W_{13}$	6
W_{26}*	26	$2, 4, \ldots, 14$	$W_{13} \oplus W_{13}$	7
W_{27}*	27	$1, 3, 5, 7, 9$	$W_{12} \oplus W_{15}$ and $W_{13} \oplus W_{14}$	5
W_{28}*	28	$0, \ldots, 12$	$W_{12} \oplus W_{16}$	7
W_{29}*	29	$1, 3, \ldots, 15$	$W_{13} \oplus W_{16}$	8
W_{30}	30	$2, 4, 6, 10$	$W_{14} \oplus W_{16}$ and $W_{12} \oplus W_{18}$	≥ 4
W_{31}	31	$1, 3, \ldots, 9$	$W_{15} \oplus W_{16}$ and $W_{13} \oplus W_{18}$	≥ 5
W_{32}	32	$0, \ldots, 12, 16$	$W_{16} \oplus W_{16}$ and $W_{12} \oplus W_{20}$	≥ 8
W_{33}*	33	$1, 3, \ldots, 15$	$W_{13} \oplus W_{20}$	8
W_{34}	34	$2, 4, \ldots, 10$	$W_{10} \oplus W_{24}$	≥ 5
W_{35}	35	$1, 3, \ldots, 13$	$W_{15} \oplus W_{20}$ and $W_{13} \oplus W_{22}$	≥ 7
W_{36}*	36	$0, \ldots, 16$	$W_{12} \oplus W_{24}$ and $W_{13} \oplus W_{23}$	9
W_{37}*	37	$1, 3, \ldots, 15$	$W_{13} \oplus W_{24}$	8
W_{38}*	38	$2, 4, \ldots, 18$	$W_{13} \oplus W_{25}$	9
W_{43}*	43	$1, \ldots, 17$	$W_{13} \oplus W_{20}$ and $W_{15} \oplus W_{28}$	9
W_{44}*	44	$0, \ldots, 20$	$W_{16} \oplus W_{28}$	11
W_{45}*	45	$1, \ldots, 23$	$W_{13} \oplus W_{32}$	
W_{47}	47	$1, \ldots, 17$	$W_{13} \oplus W_{34}$	≥ 9
W_{48}	48	$0, \ldots, 20, 24$	$W_{16} \oplus W_{32}$	≥ 12
W_{60}*	60	$0, \ldots, 28$	$W_{28} \oplus W_{32}$	15
W_{61}*	61	$1, \ldots, 31$	$W_{16} \oplus W_{45}$	16

Table 2: The SNF of some weighing matrices
* Indicates that this spectrum meets the bounds given in the theorem.

Remark 2 Table 2 gives a $W(60,9)$ with canonical SNF for $s = 0,\ldots,28$, considerably improving our previous estimate of $s = 0,\ldots,20$.

4 SNF and Projective Planes

In Seberry Wallis [**?**, p. 411] we note the following theorem:

Theorem 3 *The incidence matrix of a (v,k,λ)-configuration is equivalent to a diagonal matrix with entries*

$$\begin{cases} 1 & \frac{v+1}{2} \ times, \\ (k-\lambda) & \frac{v-3}{2} \ times, \\ k(k-\lambda) & once, \end{cases}$$

when $k-\lambda$ is squarefree and $(k-\lambda,k) = 1$. □

We apply this theorem to $SBIBD(q^2+q+1, q+1, 1)$. These always exist when q is a prime power.

Lemma 3 *The incidence matrix of a $SBIBD(q^2+q+1, q+1, 1)$ is equivalent to a diagonal matrix with entries*

$$\begin{cases} 1 & \frac{1}{2}(q^2+q+2) \ times, \\ q & \frac{1}{2}(q^2+q-2) \ times, \\ q(q+1) & once, \end{cases}$$

when q is squarefree. □

We found the following results using cyclic difference sets, D, from Ryser [**?**, p. 132].

q	SBIBD	D	SNF
2	(7,3,1)	$\{0,1,3\}$	$1^4 2^2 6$
3	(13,4,1)	$\{0,1,3,9\}$	$1^7 3^5 12$
5	(31,6,1)	$\{0,1,3,8,12,18\}$	$1^{16} 5^{14} 30$
7	(57,8,1)	$\{0,1,3,13,32,36,43,52\}$	$1^{29} 7^{27} 56$
$4 = 2^2$	(21,5,1)	$\{0,1,4,14,16\}$	$1^{10} 2^2 4^8 20$
$8 = 2^3$	(73,9,1)	$\{0,1,3,7,15,31,36,54,63\}$	$1^{28} 2^9 4^9 8^{26} 72$

Future work will be concerned with SNF of inequivalent projective planes and the cases where q is not squarefree.

5 Circulant Weighing Matrices

It is shown in Geramita and Seberry [**?**, p. 153] that if Q is the circulant incidence matrix of a projective plane of order q then $W = Q^2 - J$, where J is the matrix of all ones, is a circulant $W(q^2+q+1, q^2)$. In order to search for relationships between the SNF of the $SBIBD(q^2+q+1, q+1, 1)$ and its complement signed, which is a $W(q^2+q+1, q^2)$, we found the SNF of the corresponding weighing matrix.

$\mathbf{q}$	$\mathbf{W(q^2+q+1, q^2)}$	**SNF**
2	$W(7,4)$	$1^3 2^4 3^3$
3	$W(13,9)$	$1^3 3^7 9^3$
4	$W(21,16)$	$1^9 4^3 16^9$
5	$W(31,25)$	$1^{15} 5^1 25^{15}$
7	$W(57,49)$	$1^{10} 7^{37} 49^{10}$
8	$W(73,64)$	$1^{22} 8^{29} 64^{22}$

In addition we found the circulant $W(31,16)$ with first row

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had SNF, $1^5 2^5 4^{11} 8^5 16^5$.
As yet we have been unable to establish a theoretical link between them.

6 Further Numerical Results

Remark 3 We took the $OD(16;1,1,1,1,2,2,2,2,2)$ in Geramita and Seberry [**?**, p. 361] and used all possible substitutions for the variables to obtain a $W(16,9)$. Every one of these $W(16,9)$ had SNF, $1^8 9^8$.

Remark 4 When the variables of the $OD(16;1,1,1,1,1,1,5,5)$ were chosen using all possible combinations which gave a $W(16,9)$ all yielded the SNF, $1^8 9^8$.

Remark 5 Working with the $OD(16;1,1,1,1,2,2,4,4)$ and choosing $x_1 = x_6 = 1$, $x_2 = 0$ with $x_3 = x_4 = x_8 = 1$, $x_5 = x_7 = 0$ and with $x_3 = x_4 = x_8 = 0$, $x_5 = x_7 = 1$ both gave $W(16,9)$ with SNF, $1^8 9^8$.

Remark 6 We use the sequences $\{x_1, x_2, -x_3, 0\}$, $\{x_1, -x_2, 0, x_4\}$, $\{x_2, x_3, x_4, 0\}$, $\{x_1, 0, x_3, -x_4\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16;3,3,3,3)$. With the variables chosen as $x_1 = x_2 = x_3 = 1$, $x_4 = 0$, with $x_1 = x_2 = x_4 = 1$, $x_3 = 0$, with $x_1 = x_3 = x_4 = 1$, $x_2 = 0$, and with $x_2 = x_3 = x_4 = 1$, $x_1 = 0$ we obtain a $W(16,9)$ with SNF, $1^8 9^8$, $1^8 9^8$, $1^6 3^4 9^6$, and $1^6 3^4 9^6$ respectively.

Remark 7 We use the sequences $\{x_4, x_1, -x_4, 0\}$, $\{x_4, x_2, -x_4, 0\}$, $\{x_4, x_3, x_4, 0\}$, $\{x_4, -x_3, x_4, 0\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16;1,1,2,8)$. With the variables chosen as $x_1 = x_4 = 1, x_2 = x_3 = 0$, and with $x_1 = x_3 = 0, x_2 = x_4 = 1$ we obtain a $W(16,9)$ with the same SNF, $1^8 9^8$.

Remark 8 We use the sequences $\{x_1, 0, 0, 0\}$, $\{x_2, 0, 0, 0\}$, $\{x_3, x_3, x_4, -x_4\}$, $\{x_4, x_4, -x_3, x_3\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16;1,1,4,4)$. With the variables chosen as $x_1 = x_3 = x_4 = 1, x_2 = 0$, and with $x_1 = 0, x_2 = x_3 = x_4 = 1$ we obtain $W(16,9)$ with the same SNF, $1^8 9^8$.

Remark 9 We use the sequences $\{x_2, x_1, -x_2, 0\}$, $\{x_2, 0, x_2, 0\}$, $\{x_3, x_4, 0, 0\}$, $\{x_3, -x_4, 0, 0\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16; 1, 2, 2, 4)$. When the variables are chosen as $x_1 = x_2 = x_3 = x_4 = 1$ the $W(16, 9)$ so obtained has SNF, $1^8 9^8$.

Remark 10 We use the sequences $\{x_1, x_2, x_3, 0\}, \{x_1, x_2, -x_3, 0\}, \{-x_2, x_1, -x_2, 0\}, \{x_2, x_4, -x_2, 0\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16; 1, 2, 3, 6)$. With the variables chosen as $x_1 = x_2 = 1, x_3 = x_4 = 0$, and with $x_1 = 0, x_2 = x_3 = x_4 = 1$ we obtain $W(16, 9)$ with SNF, $1^6 3^4 9^6$, and $1^8 9^8$ respectively.

Remark 11 We use the sequences $\{x_2, 0, x_2, 0\}, \{x_2, x_1, -x_2, 0\}, \{x_3, x_3, x_4, -x_4\}, \{x_4, x_4, -x_3, x_3\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16; 1, 4, 4, 4)$. With the variables chosen as $x_1 = x_2 = x_3 = 1, x_4 = 0$, with $x_1 = x_2 = x_4 = 1, x_3 = 0$, and with $x_1 = x_3 = x_4 = 1, x_2 = 0$ we obtain a $W(16, 9)$ with the same SNF, $1^8 9^8$.

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