New Weighing Matrices

Christos Koukouvinos
Department of Mathematics
National Technical University of Athens
Zografou 157 73
Athens
Greece

and

Jennifer Seberry*
Department of Computer Science
University of Wollongong
Wollongong
NSW 2522
Australia

Abstract

New weighing matrices and skew weighing matrices are given for many orders $4t \leq 100$. These are constructed by finding new sequences with zero autocorrelation. These results enable us to determine for the first time that for $4t \leq 84$ a $W(4t,k)$ exists for all $k = 1, \ldots, 4t-1$ and also that there exists a skew-weighing matrix (also written as an $OD(4t;1,k)$) for $4t \leq 80$, t odd, $k = a^2+b^2+c^2, a,b,c$ integers except $k = 4t-2$ must be the sum of two squares.
Key words and phrases: Autocorrelation, construction, associated polynomial,sequence.
AMS Subject Classification: Primary 62K05, 62K10, Secondary 05B20

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [3], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A. It was also shown in [3] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$.

A *weighing matrix* $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n;k)$. The number k is called the *weight* of W.

*Research funded by ARC grant A48830241.

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n the *non-periodic autocorrelation function* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1, \tag{1}$$

If $A(z) = a_1 + a_2 z + \ldots + a_n z^{n-1}$ is the associated polynomial of the sequence A, then

$$A(z)A(z^{-1}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), z \neq 0. \tag{2}$$

Given A as above of length n the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \tag{3}$$

Sequences with zero periodic autocorrelation function or zero non-periodic autocorrelation function, of length t, are used to form the first rows of four circulant matrices which are then used in the Goethals-Seidel array to form matrices of order $4t$ of the required type (if orthogonal designs) or weight (if weighing matrices). In the case of sequences with zero non-periodic autocorrelation function, the sequences are first padded with sufficient zeros added to the end to make their length t.

The results used are:

Theorem 1 [5, Theorem 4.49] *If there exist four circulant matrices A_1, A_2, A_3, A_4 of order n satisfying*

$$\sum_{i=1}^{4} A_i A_i^T = fI$$

where f is the quadratic form $\sum_{j=1}^{u} s_j x_j^2$, then there is an orthogonal design $OD(4n; s_1, s_2, \ldots, s_u)$.

Corollary 1 *If there are four $\{0, \pm 1\}$-sequences of length n and weight w with zero periodic or non-periodic autocorrelation function then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form $OD(4n; w)$ or a $W(4n, w)$. If one of the sequences is skew-type then they can be used similarly to make an $OD(4n; 1, w)$. We note that if there are sequences of length n with zero non-periodic autocorrelation function then there are sequences of length $n + m$ for all $m \geq 0$.*

Weighing matrices have long been studied because of their use in weighing experiments as first studied by Hotelling [7] and later by Raghavarao [12] and others. For more applications of weighing matrices see Banerjee [1] and Harwit and Sloane [6]. There are a number of conjectures concerning weighing matrices:

Conjecture 1 *There exists a weighing matrix $W(4t, k)$ for $k \in \{1, \ldots, 4t\}$.*

Conjecture 2 *When $n \equiv 4(mod\ 8)$, there exists a skew-weighing matrix (also written as an $OD(n;1,k)$) when $k \leq n-1$, $k = a^2+b^2+c^2, a, b, c$ integers except that $n-2$ must be the sum of two squares.*

Conjecture 3 *When $n \equiv 0(mod\ 8)$, there exists a skew-weighing matrix (also written as an $OD(n;1,k)$) for all $k \leq n-1$.*

The reader in referred to Geramita and Seberry [5] for all other undefined terms.

In Geramita and Seberry [5] the status of the weighing matrix conjecture is given for $W(4t,k), k \in \{1,\ldots,4t\}$ and $t \in \{1,\ldots,21\}$. We give new results including resolving the conjecture in the affirmative for all $4t \leq 84$.

2 Some new sequences with zero autocorrelation

Tables 1 to 6 give new results which allow us to settle in the affirmative Conjecture 2 for orders 68 and 76. They also eliminate most unsolved cases for Conjecture 1 in order 92 and Conjecture 2 in order 84.

Length=19	Sequences with zero non-periodic autocorrelation function
1,62	$\{0+++-+-0+\ a-0+-+---0\}$,
	$\{+-+++--++0--+-0++++\}$,
	$\{+-+++--++0++-+0----\}$,
	$\{0+++-+-0+0+0-+-+++0\}$
1,66	$\{0+++-+-++\ a--+-+---0\}$,
	$\{0+++-+-++0++-+-+++0\}$,
	$\{+++-\ 0+---------++-+++-\ 0\}$,
	$\{+++-\ 0+---+++--+---+\ 0\}$
1,68	$\{++-+\ 0+---a+++-0-+--\}$,
	$\{+++-+++-++0++-+--+++\}$,
	$\{+++-+++-++0--+-+++--\}$,
	$\{++-+\ 0+---0---+0+-++\}$
1,72	$\{--++++++-+\ a-+-------++\}$,
	$\{+-+-+++-++0+-+++--++\}$,
	$\{+-+-+++-++0-+---+++--\}$,
	$\{--++++++-+0+-++++++--\}$

Table 1: Sequences of length 19 with zero non-periodic autocorrelation function

3 Numerical consequences

We use the tables of Appendix H and extend Theorem 4.149 of Geramita and Seberry [5].

We note that sequences with periodic autocorrelation function zero exist for $W(4t,4t-1)$, $W(4t,4t)$ for all $t \in \{1,\ldots,31\}$ [15]. We use the $(OD;1,1,64)$ for $n \geq 17$ and the $(OD;1,1,80)$ for $n \geq 21$ from [11]. Hence using the results of [8], [9], [10], [11] and those given in Tables 1, 2 and 3, we have:

Length=19	Sequences with zero periodic autocorrelation function
1,61	$\{0\ 0 - + + - + - - \ a\ + + - + - - + \ 0\ 0\}$,
	$\{+ + + + + + - - - + - + - + + + \ 0\ 0\ 0\}$,
	$\{+ + - + + + - - + + - + - + \ 0\ 0 + \ 0\ 0\}$,
	$\{- + \ 0 - - - + + - + + + + + - - - \ 0\ 0\}$
1,67	$\{+ - - - \ 0 + + - + \ a\ - + - - 0 + + + -\}$,
	$\{0 + + + + + + - + + + \ 0 + - + + - - + -\}$,
	$\{- + + + + - + - + + - - - + - + \ 0 + \ 0\}$,
	$\{- \ 0 + + - + + + - + + + + - - - - + \ 0\}$
1,69	$\{\ 0 + - - + + + - + \ a\ - + - - - + + - \ 0\}$,
	$\{- + + + + + - - - + - + - - + \ 0\ 0 + -\}$,
	$\{+ + - \ 0 + - + - - + - - - + + + + + -\}$,
	$\{- \ 0 + + + + + + + + + - + - + - + + + -\}$,
1,70	$\{+ + + + + - + - + \ a\ - + - + - - - - -\}$,
	$\{+ \ 0 - + - + + - - + - + + + - + + + +\}$,
	$\{+ + - + + + + - - \ 0 + + + - - + + - + \ 0\}$,
	$\{- + + + \ 0 - + + - + + + - \ 0 - - - + +\}$
1,1,74	$\{+ + - + - + + + - \ a\ + - - - + - + - -\}$,
	$\{- + - - + - - - - - \ b\ + + + + - + + - +\}$,
	$\{- + + - - - - - - + + + - - - - - - + + -\}$,
	$\{- + + - + - + + + + + + + - + - + + -\}$,

Table 2: Sequences of length 19 with zero periodic autocorrelation function

Theorem 2 *There exists an orthogonal design $OD(4n; 1, k)$ when*

(i) *for $n \geq t$, $t = 3, 5, 7, 9$ with $k \in \{x : x \leq 4t - 1,\ x = a^2 + b^2 + c^2\}$;*

(ii) *for $n \geq 11$, with $k \in \{x : x \leq 43,\ x = a^2 + b^2 + c^2,\ x \neq 42\}$;*

(iii) *for $n \geq 13$, with $k \in \{x : x \leq 51,\ x = a^2 + b^2 + c^2\}$;*

(iv) *for $n \geq 15$, with $k \in \{x : x \leq 59,\ x = a^2 + b^2 + c^2\}$;*

(v) *for $n \geq 17$, with $k \in \{x : x \leq 67,\ x = a^2 + b^2 + c^2,\ x \neq 61, 66\}$;*

(vi) *for $n \geq 19$, with $k \in \{x : x \leq 75,\ x = a^2 + b^2 + c^2,\ x \neq 61\}$.*

(vii) *for $n \geq 21$, with $k \in \{x : x \leq 83,\ x = a^2 + b^2 + c^2,\ x \neq 61, 77, 78, 82\}$.*

(viii) *for $n \geq 23$, with $k \in \{x : x \leq 91,\ x = a^2 + b^2 + c^2,\ x \neq 61, 77, 78, 82, 85, 86, 89, 90, 91\}$.*

(ix) *for $n \geq 25$, with $k \in \{x : x \leq 99,\ x = a^2 + b^2 + c^2,\ x \neq 61, 77, 78, 82, 85, 86, 89, 90, 91, 92, 93, 94, 97, 98, 99\}$.*

All are constructed by using four circulant matrices in the Goethals-Seidel array.

Proof. We have the results for $n \geq 13$ for all k except 46 and 49 from [11]. The $OD(52; 1, 46)$ is given in [10, Table 2]. An $OD(52; 1, 49)$ is given in [5, Theorem 8.38]. The $OD(4n; 1, 46)$ and $OD(4n; 1, 49)$ for $n \geq 15$ are given in Table 1 of [8].

The existence of $OD(4n; 1, 57)$ $n \geq 15$ and $OD(4n; 1, 67)$ for $n \geq 17$ is established in [9].

For $n \geq 17$ [11] gives the result except for $k = 46, 49, 61, 62$. The result for 62 is given in Table 1. An $OD(68;1,66)$ does not exist as its existence would imply the existence of an $OD(68;1,1,66)$ by the Geramita-Verner Theorem [5, Theorem 2.20] and the existence of that design requires that 66 should be the sum of two squares which it is not. Hence the $OD(68;1,66)$ does not exist. The value 60 was erroneously included in [11] Theorem 8: it should not have been included as 60 is a number of the form $4^a(8b+7)$ and so it cannot be written as three squares as required by the conditions of Conjecture 2.

$OD(4n;1,k)$ for $k = 70$ exist for $n = 19$ from Table 2, k = 21 from [9], and for $n \geq 23$ from Table 5.

$OD(4n;1,k)$ for $k = 68$, 72 and $n \geq 19$ are given in Table 1, the result for $k = 66$ and 68 with $n \geq 19$ is given in [5, Table H.2]. $OD(4n;1,76)$ for $n \geq 21$ is given in Table 4 and $OD(4n;1,k)$ for $k = 68$, 72 and 83 are given in [9]. The $OD(4n;1,k)$ for $k = 84$, 88 and $n \geq 23$ are given in Table 4. The $OD(4n;1,75)$ for $n = 21$ is from Table 3 and for $n \geq 23$ from Table 5.

The $OD(4n;1,k)$ for $k = 69$, 74 and 75 exist for n =19 from Table 2 and for $n \geq 21$ from [9]. The $OD(76;1,73)$ is given in [8] and the $OD(4n;1,73)$, $n \geq 21$, in [9].

The $OD(4n;1,96)$, $n \geq 25$ is given in Table 6. □

Length=21	Sequences with zero periodic autocorrelation function
1,57	{− − − − + − 0 0 0 − a + 0 0 0 + − + + + +},
	{+ + 0 0 0 + + + − + + + − + − 0 0 0 − + +},
	{+ 0 0 0 − + + + 0 + 0 − + + − − − + 0 − 0},
	{+ + − 0 − + 0 + + 0 + − − 0 0 − + 0 0 + −}
1,61	{+ + + − + 0 − + 0 + a − 0 − + 0 − + − −−},
	{+ − + − + + − − + 0 + + + − + + 0 + 0 0 0},
	{+ + + − + 0 + + + + − − + 0 − − + 0 0 0 0},
	{+ + 0 0 − 0 0 0 − + + 0 − − − − + − 0 + +}
1,62	{− − − − + 0 − 0 − 0 a 0 + 0 + 0 − + + ++},
	{+ 0 + + − + − − 0 + 0 + 0 + + + − + − + +},
	{+ + + + 0 − − + − 0 0 + − 0 + − 0 + − 0+},
	{ 0 + + − − + + 0 − 0 0 − + − + − 0 + + + −}
1,67	{ 0 − − + + + − 0 + − a + − 0 + − − − − + + 0},
	{+ 0 − + − + − + 0 + 0 + 0 + + − − + + + +},
	{+ + 0 + − + − − 0 0 − − + + + + − + + − 0},
	{ 0 0 − − − + + + + + + − + + − + − + 0 − 0}
1,73	{+ + + + + + + − + − 0 a 0 + − + − − − − − −−},
	{− + + − 0 + + + − + 0 + + + − 0 − + + +−},
	{+ + − 0 + + − + + 0 − + − + 0 − + + + + −},
	{− + − + + + − − − − + + − − + − 0 + 0 + +}
1,74	{+ + − − − − − + − + − a + − + − + + + + + − −},
	{− + + + + + + − − + 0 0 + − + + + + + + − + −},
	{+ 0 0 0 − + + + + + + + + − + − − − + − −+},
	{+ + − + − + 0 0 + + − − − + + − 0 + 0 − +}

Table 3 : Sequences of length 21 with zero periodic autocorrelation function

Length=21	Sequences with zero periodic autocorrelation function
1,75	$\{+\,+\,-\,+\,-\,+\,+\,+\,+\,0\;a\;0\,-\,-\,-\,-\,+\,-\,+\,-\,-\}$,
	$\{-\,0\,+\;\;0\,+\,+\,-\,-\,-\,+\,+\,+\,-\,+\,+\,+\,-\,+\,+\,+\,-\}$,
	$\{-\,+\,+\,+\,+\,+\,+\,-\,+\,+\,-\,-\,-\,+\,-\,+\,+\,0\;0\,+\,-\}$,
	$\{+\,+\,-\,+\,-\,+\,+\,-\,+\,+\,+\,+\,+\,-\,-\,+\,-\,+\;\;0\;0\,-\}$
1,76	$\{+\,+\,-\,-\,-\,+\,-\,+\,+\,-\;\;a\,+\,-\,-\,+\,-\,+\,+\,+\,-\,-\}$,
	$\{-\,0\,-\;\;0\,+\,+\,+\,+\,+\,+\,+\,+\,+\,-\,+\,-\,-\,+\,+\,0\,+\,-\}$,
	$\{-\,+\,+\,+\,+\,-\,+\,-\,+\,+\,-\,+\,-\,+\,-\,+\,+\,+\;\;0\,+\,-\}$,
	$\{+\,0\,-\;\;0\,+\,-\,-\,-\,+\,+\,-\,-\,-\,+\,-\,+\,+\,+\,0\,+\,+\}$
1,77	$\{-\,+\,+\,+\,+\,+\,+\,-\,+\;\;0\,-\;\;a\,+\;\;0\,-\,+\,-\,-\,-\,-\,-\,+\}$,
	$\{-\,0\,+\,+\,+\,+\,-\,+\,-\,+\,+\,+\,-\,+\,+\,-\,-\,+\,+\,+\,-\}$,
	$\{+\,+\;\;0\,+\,+\,+\;\;0\,+\,-\,+\,-\,+\,+\,-\,+\,-\,+\,+\,-\,-\,-\}$,
	$\{-\,+\,-\,+\,+\,0\,-\,-\,+\,+\,-\,-\,+\,-\,+\,+\,+\,+\,+\,+\,+\,-\}$
79	$\{+\,+\,-\,+\;\;0\,-\,-\,-\,+\,+\,-\,+\,+\,+\,+\,+\;\;0\,+\,-\,+\,+\}$,
	$\{\;0\,+\,+\,+\,-\,-\,-\,+\,+\,+\,-\,+\;\;0\,+\,+\,-\,+\,+\,-\,-\,+\}$,
	$\{-\,+\,-\,+\,+\,+\,-\,+\,+\,+\,+\,-\,-\,-\,+\,-\,-\,+\;\;0\,+\,-\}$,
	$\{+\,-\,+\,+\,-\,-\,+\,-\,-\,+\,-\,+\,-\,+\,-\,-\,-\,+\,+\,+\,+\}$
1,1,82	$\{-\,-\,+\,-\,-\,+\,+\,+\,-\,-\;\;a\,+\,+\,-\,-\,-\,+\,+\,-\,+\,+\}$,
	$\{+\,+\,+\,+\,-\,-\,-\,+\,-\,-\;\;b\,+\,+\,-\,+\,+\,+\,-\,-\,-\,-\}$,
	$\{-\,+\,-\,+\,+\,-\,+\,-\,+\,-\,+\,-\,+\,-\,+\,-\,+\,+\,-\,+\,-\}$,
	$\{-\,+\,-\,-\,+\,+\,+\,+\,+\,+\,+\,+\,+\,+\,+\,+\,+\,-\,-\,+\,-\}$

Table 3 (cont): Sequences of length 21 with zero periodic autocorrelation function

Length=21	Sequences with zero non-periodic autocorrelation function
1,68	$\{+\,+\;\;0\,+\,+\,-\,+\,0\,+\,-\;a\;\;+\,-\,0\,-\,+\,-\,-\,0\,-\,-\}$,
	$\{-\,+\,+\,-\,+\,+\,+\,0\,-\,-\,0\,+\,+\,0\,-\,-\,-\,+\,+\,-+\}$,
	$\{-\,+\,+\,-\,+\,+\,+\,0\,-\,-\,0\,-\,-\,0\,+\,+\,+\,-\,-\,+-\}$,
	$\{+\,+\;\;0\,+\,+\,-\,+\,0\,+\,-\,0\,-\,+\,0\,+\,-\,+\,+\,0\,+\,+\}$
1,72	$\{\;0\,+\,-\,+\,+\,-\,-\,+\,+\,+\;\;a\;\;-\,-\,-\,+\,+\,-\,-\,+\,-\;\;0\}$,
	$\{+\,0\,+\,-\,+\,-\,+\,+\,-\,+\,0\,-\,-\,-\,+\,+\,+\,+\,+\,0+\}$,
	$\{+\,0\,+\,-\,+\,-\,+\,+\,-\,+\,0\,+\,+\,+\,-\,-\,-\,-\,-\,0-\}$,
	$\{0\,+\,-\,+\,+\,-\,-\,+\,+\,+\;\;0\,+\,+\,+\,-\,-\,+\,+\,-\,+\;\;0\}$
74	$\{-\,+\;\;0\,+\,+\,+\,+\,0\,+\,+\,0\,-\,+\,-\,+\,0\,-\,-\,+\,++\}$,
	$\{-\,+\;\;0\,+\,+\,+\,+\,0\,+\,+\,0\,+\,-\,+\,-\,0\,+\,+\,-\,-\,-\}$,
	$\{-\,+\,-\,+\,+\,-\,-\,+\,+\,-\;\;0\,+\,+\,-\,-\,-\,-\,+\,-\,+\,+\}$,
	$\{-\,+\,-\,+\,+\,-\,-\,+\,+\,-\;\;0\,-\,-\,+\,+\,+\,+\,-\,+\,-\,-\}$
1,76	$\{+\,+\,+\,0\,-\,-\,+\,-\,-\,+\,a\;\;-\,+\,+\,-\,+\,+\,0\,-\,-\,-\}$,
	$\{-\,+\,+\,+\,+\,-\,+\,+\,+\,+\;\;0\,+\,+\,-\,-\,+\,-\,+\,-\,+\,-\}$,
	$\{-\,+\,+\,+\,+\,-\,+\,+\,+\,+\;\;0\,-\,-\,+\,+\,-\,+\,-\,+\,-\,+\}$,
	$\{+\,+\,+\,0\,-\,-\,+\,-\,-\,+\,0\,+\,-\,-\,+\,-\,-\,0\,+\,++\}$

Table 4: Sequences of length 21 with zero non-periodic autocorrelation function

Length=23	Sequences with zero non-periodic autocorrelation function
1,70	$\{0-+0++++--+\ a\ -++----\ 0-+0\}$,
	$\{-++0+\ 0+\ 0+-+\ 0-++\ 0-++\ 0+--\}$,
	$\{-++0+\ 0+\ 0+-+\ 0+--\ 0+--\ 0-++\}$,
	$\{0-+0++++--+\ 0+--++++\ 0+-0\}$
1,74	$\{-0-\ 0\ 0++-++-a\ +--+--0\ 0+\ 0+\}$,
	$\{-++++0+++-+\ 0-+-+++---++\}$,
	$\{-++++0+++-+\ 0+-+---+++--\}$,
	$\{-0-\ 0\ 0++-++-0-++-++0\ 0-\ 0-\}$
1,84	$\{+-++-+++---a\ +++---+--+-\}$,
	$\{++-++0-++++\ 0-+-+-\ 0++++-\}$,
	$\{++-++0-++++\ 0+-+-+\ 0----+\}$,
	$\{+-++-+++---0---+++-++-+\}$
1,88	$\{-++--+-++++a\ ----+-++--+\}$,
	$\{+--+++-++++0-+-+-++++--\}$,
	$\{+--+++-++++0+-+-+----++\}$,
	$\{-++--+-++++0++++-+--++-\}$

Table 5: Sequences of length 23 with zero non-periodic autocorrelation function

Length=25	Sequences with zero non-periodic autocorrelation function
1,96	$\{+-++-++++---\ a\ +++----+--+-\}$,
	$\{+++-+-+++-++0+---++-+++-+\}$,
	$\{+++-+-+++-++0-+++--+---+-\}$,
	$\{+-++-++++---0---++++-++-+\}$,

Table 6: Sequences of length 25 with zero non-periodic autocorrelation function

Theorem 3 *There exists a $W(4n;k)$ when*

(i) *for $n \geq t$, $t = 3, 5, 7, 9, 11, 13, 15, 17, 19$ with $k \in \{x : x \leq 4t\}$;*

(ii) *for $n \geq 21$, with $k = 1, \ldots, 78, 80, 81, 82$;*

(iii) *for $n \geq 23$, with $k = 1, \ldots, 78, 80, 81, 82, 85, 86, 88, 89, 90, 92$.*

All are constructed by using four circulant matrices in the Goethals-Seidel array.

Proof. We use Theorem 2, the $W(4t, 4t)$ and $W(4t, 4t-1)$ from [15] and Theorem 3 from [11].

The result for 75 in (i) comes from Theorem 1, Tables 3 and 5 and for 77, in (ii) from Theorem 1 and from Table 4. The result for 78, 80, 81, 82 and 84 in (ii) comes from [11, Table 8] as does the result for 86, 88, 90 and 92 in (iii). Table 5 gives 71, 85 and 89 of (iii). □

Lemma 1 *The necessary conditions are sufficient for the existence of $OD(4n; 1, k)$ for $n = 3, 5, \ldots, 19$ and $k \leq 4n-1$. All are constructed from four circulant matrices in the Goethals-Seidel array.*

Proof. Use Theorem 2 for all the results for n $\leq$ 11.

The $OD(52;1,46)$ and $OD(60;1,57)$ required are given in Tables 2 and 4 of [10].

The $OD(68;1,k)$ for k = 57, 61, and 67 are given in [11, Table 6] and [10, Table 6]. An $OD(68;1,66)$ does not exist as was shown in the proof of Theorem 2 above. The value 60 was erroneously included in [11]: an $OD(68;1,60)$ does not exist.

The $OD(76;1,k)$ for k = 57, 61, 66, 67, 69, 70, 73, 74 and 75 are given in Tables 2 and 3 of [8], and Table 2 of this paper. This gives us the result of the enunciation. □

Lemma 2 *There exists an* $OD(84;1,k)$ *for* $k \in \{x : x \leq 83,\ x = a^2 + b^2 + c^2\}$ *with the possible exception of* 70, 78 *which are undecided. All may be constructed using four circulant matrices in the Goethals-Seidel array.*

Proof. The results for k = 57, 61, 67, 73, 75, 77, 82 and 83 which may be constructed using four circulants in the Goethals-Seidel array are given in Table 3 of this paper. □

Lemma 3 *There exists an* $OD(92;1,k)$ *for* $n \geq 23$, *with* $k \in \{x : x \leq 91,\ x = a^2 + b^2 + c^2\}$ *with the possible exception of* 61, 77, 78, 82, 85, 86, 89, 90, 91 *which are undecided.*

Proof. Use Theorem 2. □

Lemma 4 *There exists a* $W(4n,k)$ *for* $k \in \{x : 0 \leq x \leq 4n\}$ *with* n = 1, 3, ..., 21. *All are constructed from four circulant matrices in the Goethals-Seidel array.*

Proof. Follows from [11], Lemma 1 and Tables 1 and 2. $k = 71$ is given in [9]. The sequences for $W(84,79)$ and $W(84,77)$ are given in Tables 3 and 4. □

Lemma 5 *There exists a* $W(92,k)$ *for all* k *except possibly* 79, 87, *which are undecided. All are constructed by using four circulant matrices in the Goethals-Seidel array.*

Proof. The paper [11] gives all k except possibly 71, 73, 75, 77, 79, 83, 85, 87 and 89. Koukouvinos [9] gives $k = 83$. The sequences for 71, 73, 75, 77, 85, and 89 are given in Tables 2, 4 and 5. □

Lemma 6 *There exists a* $W(100,k)$ *for all* k *except possibly* 87, 91, 93, 95 *which are undecided. All are constructed by using four circulant matrices in the Goethals-Seidel array.*

Proof. The paper [11] gives all k except possibly 71, 77, 83, 87, 89, 91, 93, 95, 97. Koukouvinos [9] gives $k = 83$. The sequences for 71, 77 and 89 are given in Tables 4 and 5. The $OD(100;1,96)$ in Table 6 gives $k = 97$.

Order	Applicable Conjecture	Unresolved Cases	Applicable Conjecture	Unresolved Cases
4	1	true	2	true
8	1	true	3	true
12	1	true	2	true
16	1	true	3	true
20	1	true	2	true
24	1	true	3	true
28	1	true	2	true
32	1	true	3	true
36	1	true	2	true
40	1	true	3	true
44	1	true	2	true*
48	1	true	3	true
52	1	true	2	true
56	1	true	3	true
60	1	true	2	true
64	1	true	3	true
68	1	true	2	true*
72	1	true	3	true
76	1	true	2	true
80	1	true	3	true
84	1	true	2	70,78
88	1	true	3	true
92	1	79,87	2	61,77,78,82 85,86,89,90,91
96	1	true	3	true
100	1	87,91,93,95	2	61,77,78,82,85, 86,89,90,91,92, 93,94,97,98
104	1	95	3	94,95
112	1	true	3	true
120	1	true	3	true

Table 7: Summary of the Conjectures.
True signifies the conjecture is verified.
$*OD(n; 1, n-2)$ is not possible as $n-2$ is not the sum of two squares.

References

[1] K. S. Banerjee, *Weighing Designs for Chemistry, Medicine, Economics, Operations Research and Statistics*, Marcel Dekker, New York, 1975.

[2] P. Eades and J. Seberry Wallis, An infinite family of skew-weighing matrices, *Combinatorial Mathematics IV*, in Lecture Notes in Mathematics, Vol 560, Springer-Verlag, Berlin- Heidelberg-New York, 1976, 27–60.

[3] A.V.Geramita, J.M.Geramita, and J.Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra,* 3 (1976), 281-306.

[4] A. V. Geramita, N. J. Pullman and J. Seberry Wallis, Family of weighing matrices, *Bull. Austral. Math. Soc.,* 10, (1974), 119–122.

[5] A. V. Geramita and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[6] M. Harwit and N. J. A. Sloane, *Hadamard Transform Optics*, Academic Press, New York, 1979.

[7] H. Hotelling, Some improvements in weighing and other experimental techniques, *Ann. Math. Stat.,* 15, (1944), 297–306.

[8] C. Koukouvinos, Construction of some new weighing matrices, *Utilitas Math.*, 44, (1993), 51–55.

[9] C. Koukouvinos, Some new weighing matrices, *Utilitas Math.*, 46, (1994), 81–89.

[10] C. Koukouvinos and J. Seberry, Some new weighing matrices using sequences with zero autocorrelation function, *Austral. J. Combin.*, 8, (1993), 143–152.

[11] C. Koukouvinos and J. Seberry, On weighing matrices, *Utilitas Math.*, 43, (1993), 101–127.

[12] D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, J. Wiley and Sons, New York, 1971.

[13] Jennifer Seberry, An infinite family of skew–weighing matrices, *Ars Combinatoria*, 10, (1980), 323–329.

[14] Jennifer Seberry, The skew-weighing matrix conjecture, *University of Indore Research J. Science*, 7, (1982), 1–7.

[15] Jennifer Seberry and Mieko Yamada, Hadamard matrices, sequences and block designs, in *Contemporary Design Theory - a Collection of Surveys*, eds J. Dinitz and D.R. Stinson, John Wiley and Sons, New York, (1992), 431–560.

[16] Jennifer Seberry Wallis, Hadamard matrices, Part IV, *Combinatorics: Room Squares, sum free sets and Hadamard Matrices*, Lecture Notes in Mathematics, Vol 292, eds. W. D. Wallis, Anne Penfold Street and Jennifer Seberry Wallis, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[17] J. (Seberry) Wallis, Orthogonal $(0, 1, -1)$-matrices, *Proceedings of the First Australian Conference on Combinatorial Mathematics*, (ed Jennifer Wallis and W. D. Wallis), TUNRA Ltd, Newcastle, Australia, (1972), 61-84.