Bose's Method of Differences Applied to Construct Bhaskar Rao Designs

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Abstract

In this paper we show that $BIBD(v,b,r,k,\lambda)$, where $v = pq$ or $pq+1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p,b',r',k,\lambda;G)$ where G is a group of order q and vice versa.

This gives many new GBRDs including a $GBRD(9,5,5;Z_5)$ and a $GBRD(13,7,7;Z_7)$.

1 Introduction

Let $G = \{h_1 = e,\ h_2,\ \ldots\ ,\ h_g\}$ be a finite group of order g with identity e. Form the matrix W

$$W = h_1A_1 + \ \ldots\ + \ h_gA_g$$

where $A_1,\ \ldots\ ,\ A_g$ are $v \times b$ (0,1)-matrices such that the Hadamard product $A_k * A_j = 0$ for any $k \neq j$. Let

$$W^+ = (h_1^{-1}A_1 + \ \ldots\ + \ h_g^{-1}A_g)^T$$

and

$$N = A_1 + A_2 + \ \ldots\ + \ A_g.$$

Then we say W is a *generalized Bhaskar Rao design* over G denoted by $GBRD(v,b,r,k,\lambda;G)$, or in abbreviated form $GBRD(v,k,\lambda;G)$, if N satisfies

$$NN^T = rI \ + \ \lambda J :$$

that is, N is the incidence matrix of the $BIBD(v,k,\lambda) <$ and

$$WW^+ = reI \ + \ (\lambda/g)(h_1 \ + \ \ldots\ + \ h_g)(J - I).$$

We say that the design W is based on the matrix N or that N is signed over the group G.

$GBRD$ have been studied by the author and others, for example see [7, 8, 9, 14, 15, 19, 20].

Example 1 To illustrate how a design developed from blocks using differences can be shown to be a $BIBD$ or $GBRD$ we consider the initial blocks

$$(0_2\ 1_2\ 2_2)\ \ (1_1\ 4_1\ 0_2)\ \ (2_1\ 3_1\ 0_2)\ \ (1_1\ 4_1\ 2_2)\ \ (2_1\ 3_1\ 2_2)\ \ (0_1\ 0_2\ 2_2)\ \ mod(5,2)$$

to develop the subscripts we note

$$2^{-1} = 2,\ \ 1^{-1} = 1,\ \ 2 \cdot 1^{-1} = 1 \cdot 2^{-1} = 2,\ \ 2 \cdot 2^{-1} = 1 \cdot 1^{-1} = 1$$

Differences of $(x_a\ y_b\ z_c)$ are $(y-x)_{ba^{-1}}$, $(x-y)_{ab^{-1}}$, $(z-x)_{ca^{-1}}$, $(x-z)_{ac^{-1}}$, $(z-y)_{cb^{-1}}$ and $(y-z)_{bc^{-1}}$. So the differences are

$$\begin{array}{cccccc} 1_1 & 2_1 & 1_1 & 4_1 & 3_1 & 4_1 \\ 3_1 & 4_2 & 1_2 & 2_1 & 1_2 & 4_2 \\ 1_1 & 3_2 & 2_2 & 4_1 & 2_2 & 3_2 \\ 3_1 & 1_2 & 3_2 & 2_1 & 4_2 & 2_2 \\ 1_1 & 0_2 & 4_2 & 4_1 & 0_2 & 1_2 \\ 0_2 & 2_2 & 2_1 & 0_2 & 3_2 & 3_1 \end{array}$$

This means each of the differences $0_2, 1_1, 1_1, 2_1, 2_2, 3_1, 3_2, 4_1, 4_2$ occurs 4 times. Hence we have after developing the initial blocks modulo 5, exactly Bose's example (iv) [4, p370] for a $BIBD(10,30,9,3,2)$. It is not a $GBRD$ as the difference 0_2 occurs 4 times. However it can be made into a $GRBD(6,3,4;Z_2)$ by choosing the initial blocks

$(0_2\ 1_2\ 2_2)\ \ (1_1\ 4_1\ 0_2)\ \ (2_1\ 3_1\ 0_2)\ \ (1_1\ 4_1\ 2_2)\ \ (\infty\ 2_1\ 3_1)\ \ (\infty\ 2_2\ 3_1)\ \ (\infty\ 0_1\ 2_2)\ \ (\infty\ 0_2\ 2_2)\ \ mod(5,2)$

Example 2 Consider the initial blocks

$$\begin{array}{ccccc} (1_1\ 6_1\ 0_2) & (2_1\ 5_1\ 0_2) & (3_1\ 4_1\ 0_2) & (1_2\ 6_2\ 0_3) & (2_2\ 5_2\ 0_3) \\ (3_2\ 4_2\ 0_3) & (1_3\ 6_3\ 0_1) & (2_3\ 5_3\ 0_1) & (3_3\ 4_3\ 0_1) & (0_1\ 0_2\ 0_3) \end{array}$$

$mod(7,3)$ which Bose gives as the initial blocks of a $BIBD(21,3,1)$. Without the last block of $(0_1\ 0_2\ 0_3)$ we have a $GBRD(7,3,9;Z_3)$.

Example 3 The subscripts from Bose's [4, p373, example (i)] $BIBD(9,3,1)$ give a $GBRD(3,3,9;Z_3)$ (the $GBRD$ is the 3×9 array):

$$\left[\begin{array}{ccc|ccc|ccc} 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 1 & 3 \end{array}\right]$$

Clearly, the differences from a single starting block also suffice to give all the differences, and hence we have a $GBRD(3,3,3;Z_3)$.

David Glynn [12] has found the only $GW(v,k,G)$ known to the author where G is not an abelian group. Consider the multiplication table for S_3

	1	2	3	4	5	6	
1	1	2	3	4	5	6	$1 \leftrightarrow e$
2	2	3	1	5	6	4	$2 \leftrightarrow (123)(456)$
3	3	1	2	6	4	5	$3 \leftrightarrow (132)(465)$
4	4	6	5	1	3	2	$4 \leftrightarrow (14)(26)(35)$
5	5	4	6	2	1	3	$5 \leftrightarrow (15)(24)(36)$
6	6	5	4	3	2	1	$6 \leftrightarrow (16)(25)(34)$

Then the circulant matrix

$$\begin{bmatrix}
0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 \\
0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 \\
4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 \\
0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 \\
6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 \\
5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 \\
6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 \\
1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 \\
1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 \\
0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 \\
4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 \\
1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 \\
5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0
\end{bmatrix}$$

is a generalized weighing matrix $GW(13,9;S_3)$.

There are two inequivalent $GW(13,9;Z_2)$ (see Seberry and Wehrhahn [21]) but the existence of $GW(13,9;G)$ for $G = Z_3$ or Z_6 is not yet resolved.

In this paper we show that $BIBD(v,b,r,k,\lambda)$, where $v = pq$ or $pq+1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p,b',r',k,\lambda;G)$ where G is a group of order q.

For example the blocks

$$(1_1,4_1,0_2),(2_1,3_1,0_2),(0_1,0_2,0_3)$$

developed $mod(5,3)$ give a $BIBD(15,3,1)$: the first two blocks developed, $mod(5)$ give a $GBRD(5, 3, 3; Z_3)$. The converse, that the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [18]).

Denniston [10] has given a $BIBD(66,143,13,6,1)$ design. This design is rewritten in a convenient way as follows:

$$\begin{array}{ll}
(0_3,\ 2_3,\ 3_0,\ 5_2,\ 6_2,\ 10_0), & (0_3,\ 1_4,\ 4_0,\ 6_0,\ 7_1,\ 11_2), \\
(0_4,\ 1_4,\ 3_2,\ 9_2,\ 11_1,\ 12_2), & (0_3,\ 3_4,\ 5_1,\ 7_0,\ 8_2,\ 12_1), \\
(0_4,\ 1_0,\ 3_4,\ 7_2,\ 9_0,\ 10_0), & (2_3,\ 4_4,\ 5_3,\ 6_3,\ 10_4,\ 12_4), \\
(0_4,\ 1_1,\ 3_1,\ 4_1,\ 8_0,\ 9_4), & (0_3,\ 2_0,\ 4_1,\ 5_0,\ 6_3,\ 9_1), \\
(0_3,\ 1_2,\ 2_1,\ 5_3,\ 6_1,\ 12_2), & (0_3,\ 2_2,\ 8_1,\ 9_4,\ 10_2,\ 11_0), \\
(\infty,\ 0_0,\ 0_1,\ 0_2,\ 0_3,\ 0_4), &
\end{array}$$

all blocks developed $mod(13,5)$. We note that the first 10 of these blocks also give a $GBRD(13,6,25; Z_5)$. If we knew how to sign the blocks {2 5 6 7 8 11} and {1 3 4 9 10 12}, which develop $mod(13)$ to give a $BIBD(13,6,5)$, we would have a $GBRD(13,6,5;Z_5)$.

Lemma 1 *A $GBRD(v,k,\lambda;G)$, where $|\ G\ |$ is λ, is equivalent to a $BIBD(v\lambda,k,1)$ when both are written using starting blocks developed $mod(\lambda)$.*

To illustrate we note for $k = 3$, the initial blocks

$$(x_{a_i}^{(i)}\ y_{b_i}^{(i)}\ z_{c_i}^{(i)})$$

for the $GBRD$ are equivalent to the starting blocks

$$(x_{a_i}^{(i)}\ y_{b_i}^{(i)}\ z_{c_i}^{(i)})\ \ (x_{a_i+1}^{(i)}\ y_{b_i+1}^{(i)}\ z_{c_i+1}^{(i)})\ \ (x_{a_i+2}^{(i)}\ y_{b_i+2}^{(i)}\ z_{c_i+2}^{(i)})\ \ (0_0\ 0_1\ 0_2)$$

for the $BIBD$.

When we have a $BIBD((k-1)v+1,k,k-1)$ developed $mod(v,k-1)$ with a special element ∞ we seek to form a $GBRD(v,k,(k-1)^2;G)$ where G is a group of order $k-1$. Again the converse, that the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [18]).

Lemma 2 *If the initial blocks*

$$(x_{a_1}^1\ x_{a_2}^2\ \ldots\ x_{a_k}^k)\ \ ((y_{b_1}^1\ y_{b_2}^2\ \ldots\ y_{b_k}^k)\ldots(z_{c_1}^1\ z_{c_2}^2\ \ldots\ z_{c_k}^k)$$

give a $GBRD(v,k,k-1;G)$ *where* $|\ G\ | = k-1$, *then the initial blocks*

$$(x_{a_1+i}^1\ x_{a_2+i}^2\ldots x_{a_k+i}^k)\ \ (y_{b_1+i}^1\ y_{b_2+i}^2\ldots y_{b_k+i}^k)\ldots(z_{c_1+i}^1\ z_{c_2+i}^2\ldots z_{c_k+i}^k)\ \ (\infty\ 0_1\ 0_2\ \ldots\ 0_{k-1})$$

give a $BIBD((k-1)v+1,k,1)$, *where* $i = 0,1,\ldots,k-1$. *The converse is also true.*

Comment. In some sense Bose's theorem implies that the subscripts are from a cyclic group. However the formulation in terms of $GBRD$ indicates clearly that any group will suffice.

The construction with non-cyclic groups can be visualized by replacing the 0 element of the $GBRD$ by a $(k-1)\times(k-1)$ zero matrix and the other elements by their right (left) regular matrix representation. Finally, the extra $k-1$ blocks corresponding to $(\infty\ 0_1\ 0_2\ldots 0_{k-1})$ are added.

Another interesting possibility may occur if there are parallel classes associated with a subdesign. For example the $GBRD(13,9,6;S_3)$ given by developing the block

$$(1_5\ 2_1\ 3_4\ 6_1\ 7_1\ 8_6\ 9_5\ 10_6\ 12_4)$$

$mod(13)$ can be embedded in an $SBIBD(91,10,1)$ which also has an $SBIBD(13,4,1)$ embedded (see the construction in [18]).

2 Examples and Constructions

We now give some examples of these methods. The $GBRD$ are new.

k=3: $BIBD(15,3,1)$ and $GBRD(5,3,3;Z_3)$ illustrated above. This method can be applied to $BIBD(3v,3,1)$ and $GBRD(v,3,3;Z_3)$ for all odd v.

k=4: The initial blocks for the $BIBD(28,4,1)$ are

$$[(21)_0,(01)_0,(12)_1,(10)_1],\ \ [(20)_0,(02)_0,(22)_1,(00)_1]\ \text{ and }\ [\infty,(11)_0,(11)_1,(11)_2].$$

The first two blocks give a $GBRD(9,4,3;Z_3)$ which is new.

The elements work modulo (3,3) and suffixes modulo 3.

k=5: The initial blocks of a $BIBD(45,5,1)$ are

$$\begin{array}{rl} & [(01)_0,(02)_0,(10)_2,(20)_2,(00)_1],\\ & [(21)_0,(12)_0,(22)_2,(11)_2,(00)_1]\\ \text{and} & [(00)_0,(00)_1,(00)_2,(00)_3,(00)_4].\end{array}$$

The first two blocks give a $GBRD(9,5,5;Z_5)$ which is new.

k=7: The following three sets of initial blocks from Bhat-Nayak and Kane [1, 2] give initial blocks for $BIBD(91,7,1)$:

Solution 1:

$$\begin{array}{ll} & [6_0, 7_0, 11_1, 2_1, 8_5, 5_5, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

Solution 2:

$$\begin{array}{ll} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_3, 3_3, 9_4, 4_4, 12_6, 1_6, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

Solution 3:

$$\begin{array}{ll} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

In each case the first two blocks $mod(13,7)$ are initial blocks for a $GBRD(13,7,7;Z_7)$. These are new.

One typical result illustrates many possible theorems arising from constructions for $BIBD$ using Bose's method of differences.

Lemma 3 *The starting blocks*

$$(1_1\ 2t_1\ 0_2)\ \ (2_1\ (2t-1)_1\ 0_2)\ \ \ldots\ \ (t_1\ (t+1)_1)\ 0_2)\ \ \mathrm{mod}\ (2t+1,3)$$

used by Bose [4, p373, equation 4.22] to form a $BIBD(3(2t+1),3,1)$ are also the starting blocks of a $GBRD(2t+1,3,3;Z_3)$.

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